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Strong solitary internal waves in a 2.5-layer model

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A theoretical model for internal solitary waves for stratification consisting of two layers of incompressible fluid with a constant Brunt–Väisälä frequency and a density jump at the boundary between layers ('2.5-layer model') is presented. The equation of motion for solitary waves in the case of a constant Brunt–Väisälä frequency N is linear, and nonlinearity appears due only to boundary conditions between layers. This allows one to obtain in the case of long waves a single ordinary differential equation for an internal solitary wave profile. In the case of nearly homogeneous layers the solitons obtained here coincide with the solitons calculated by Choi & Camassa (1999), and in the weakly nonlinear case they reduce to KdV solitons. In the general situation strong 2.5-layer solitons can correspond to higher modes. Sufficiently strong solitons could also possess a recirculating core (at least, as a formal solution).

The model was applied to the data collected during the COPE experiment. The results are in reasonable agreement with experimental data.

1. Introduction

'Weak' solitary internal waves (which we will also be calling solitons) with amplitudes significantly less than a characteristic vertical scale of stratification, depending on the situation, are adequately described by KdV, modified KdV (CombKdV), Benjamin–Ono, or Joseph equations (Ostrovsky & Stepanyants 1989). 'Strong' solitons with amplitudes of the order of or larger than the characteristic vertical scale of stratification are often observed experimentally and are also of interest. Strong solitons have been investigated in a number of studies (Benney & Ko 1978; Amick & Turner 1986; Turner & Vanden-Broeck 1988; Evans & Ford 1996; Derzho & Grimshaw 1997; Torez & Knio 1998; Brown & Christie 1998; Grue *et al.* 1999) and in a recent work by Choi & Camassa (1999), in particular. It seems, however, that analytical results were, in most cases, obtained for the model of two layers of homogeneous fluid with different densities or for stratifications close to constant Brunt–Väisälä frequency profile. In geophysical applications, fluid layers often have a density gradient and could be better described by layers with different Brunt–Väisälä frequencies that are constant within the layers. Fluid motion in such a system is not potential; however, similar to potential surface gravity waves, it is also described by a linear equation, and nonlinearity arises from boundary conditions between layers only. This allows one, in different limiting cases, to express an equation describing such solitons in terms of displacement of the boundary between layers only.

We use the somewhat loose term '2.5-layered fluid' here to describe a model consisting of two layers of incompressible fluid with constant Brunt–Väisälä frequencies

N_1 and N_2 and a finite density jump $\Delta\rho$ at the boundary between layers. This model can be considered as an appropriate limiting case of a three-layer fluid when the width of the intermediate layer d tends to zero, while the product $g\Delta\rho \sim N^2 d$ remains finite. In summary, equilibrium density stratification $\rho_0(z)$ is assumed here to be as follows:

$$\rho_0(z) = \begin{cases} \rho_2 - N_2^2 z/g, & 0 < z < H_2 \\ \rho_1 - N_1^2 z/g, & H_1 < z < 0, \end{cases} \tag{1.1}$$

where $\rho_1, \rho_2 \approx 1$, and $H_1 < 0$ and $H_2 > 0$ represent thicknesses of the lower and upper layers, respectively. For simplicity, density is made dimensionless by normalizing it by a constant reference value $\rho_{00} = 1\,\mathrm{g\,cm^{-3}}$.

2. Governing equations

Let us consider a stationary case when all quantities characterizing the motion are functions of $x-ct$, where c is a constant propagation speed, and vertical coordinate z. Moreover, we will be studying solutions corresponding to solitary waves only, when all perturbations tend to zero at infinity. A reduction of the equations of motion for this case was obtained first by Dubreil-Jacotin (1937) and later also by Long (1953). The result (which can be easily checked by direct substitution) is as follows. Let Ψ be a stream function defining velocities according to the relations: $v_x = -\Psi_z$, $v_z = \Psi_x$. Here the solitary waves boundary condition $\Psi \to 0, |x| \to \infty$ is assumed to be fulfilled. Then

$$\rho = \rho_0\left(z + \frac{\Psi}{c}\right), \tag{2.1}$$

where $\rho_0(z)$ is the equilibrium density profile: $\rho \to \rho_0(z)$, $|x| \to \infty$. The equation of two-dimensional motion of inviscid incompressible fluid taken in the Boussinesq approximation (i.e. density variations are taken into account only in the buoyancy term) is

$$\Psi_{xx} + \Psi_{zz} + \frac{\Psi}{c^2} N_0^2\left(z + \frac{\Psi}{c}\right) = 0, \tag{2.2}$$

where

$$N_0^2(z) = -g\rho_0'(z) \tag{2.3}$$

is a Brunt–Väisälä frequency. An expression for pressure p normalized by ρ_{00} (so that the dimension of p here is the square of the velocity) is

$$p = -c\Psi_z - \frac{\Psi_x^2 + \Psi_z^2}{2} - \int_z^{z+\Psi/c} (\xi - z) N_0^2(\xi)\,\mathrm{d}\xi - \int_0^z g\rho_0(\xi)\,\mathrm{d}\xi. \tag{2.4}$$

The kinematic boundary condition at the layer boundary $z = h(x)$ is

$$h + \frac{1}{c}\Psi(x, h) = 0. \tag{2.5}$$

By differentiating this equation, we find

$$h_x = \frac{\Psi_x}{-c - \Psi_z}. \tag{2.6}$$

Thus, the total velocity vector at the boundary in the frame of reference moving with the soliton speed is tangent to the boundary profile, as it should be in the stationary case. Similarly, at the other boundary $z = H$ (ocean bottom for the first layer and ocean surface for the second layer)

$$\Psi|_{z=H} = 0. \tag{2.7}$$

Using (2.5) and (2.6) in (2.4), we obtain the following expression for the pressure at the boundary:

$$p = -\tfrac{1}{2}(1 + h_x^2)(c + \Psi_z)^2 - g\rho h + \tfrac{1}{2}c^2, \tag{2.8}$$

where $\rho = \rho_0(0)$. This expression can be applied to both layers. Continuity of pressure at the boundary between layers gives

$$-\tfrac{1}{2}(1 + h_x^2)(c + \Psi_{1z})^2 + \tfrac{1}{2}(1 + h_x^2)(c + \Psi_{2z})^2 - g\Delta\rho h = 0, \tag{2.9}$$

where

$$\Delta\rho = \rho_1 - \rho_2 \ll 1 \tag{2.10}$$

and Ψ_1 and Ψ_2 correspond to stream functions in the first (lower) and the second (upper) layers. Derivatives Ψ_{1z} and Ψ_{2z} in (2.9) should be calculated at the boundary points $z = h(x)$.

In what follows we will be considering the case when Brunt–Väisälä frequency is constant within a given layer: $N_0(z) = N = \text{const}$. Then Ψ_{1z} and Ψ_{2z} are functionals of the solitary wave profile $h(x)$, which are determined through the solution of the linear (Helmholtz) equation

$$\Psi_{xx} + \Psi_{zz} + \frac{N^2}{c^2}\Psi = 0 \tag{2.11}$$

subject to boundary conditions (2.5) and (2.7). When Ψ_{1z} and Ψ_{2z} are expressed in terms of h, (2.9) becomes a basic equation describing a soliton profile for the 2.5-layer model. In the next section we will consider the case of long waves, allowing a simple approximate analytical treatment.

3. Long-wave limit

The solution of the linear equation (2.11) with boundary conditions (2.5) and (2.7) can be easily calculated in the limit of long waves. Let soliton slope $\varepsilon = |h_0|/w_x \ll 1$, where h_0 is soliton amplitude and w_x is the spatial width of the soliton in the horizontal direction. In this case the term Ψ_{xx} in (2.11) is small, $O(\varepsilon^2)$, and this equation can be readily solved by iterations:

$$\Psi(x,z) = -c\frac{h}{\sin\theta}\sin\left[\frac{N}{c}(z-H)\right] + \frac{c^3}{2N^2}\left(\frac{h}{\sin\theta}\right)_{xx} \times\left\{-\frac{N}{c}(z-H)\cos\left[\frac{N}{c}(z-H)\right] + \theta\cot\theta\sin\left[\frac{N}{c}(z-H)\right]\right\} + o(\varepsilon^2), \tag{3.1}$$

where

$$\theta = \theta(x) = \frac{N}{c}(h(x) - H). \tag{3.2}$$

The first term in (3.1) appears to be a solution of (2.11) with Ψ_{xx} term neglected, and the second term is a result of the first iteration. Boundary conditions (2.5) and (2.7) are obviously satisfied. From (3.1) we find

$$\Psi_z|_{z=h} = -N\cot\theta h + \frac{c^2}{2N}\left(\frac{h}{\sin\theta}\right)_{xx}\left(\frac{\theta}{\sin\theta} - \cos\theta\right). \tag{3.3}$$

Here, N and H could correspond to either layer. If expression (3.3) is substituted into (2.9) we obtain to the accuracy of $O(\varepsilon^2)$ the following equation describing soliton profile in a 2.5-layer model:

$$A(h)h_{xx} + B(h)h_x^2 + C(h) = 0, \tag{3.4}$$

where expressions for coefficients A, B, and C are as follows:

$$A(h) = -\frac{c^3}{2N_1}\left(1 - \cot\theta_1\frac{N_1}{c}h\right)^2\left(\frac{\theta_1}{\sin^2\theta_1} - \cot\theta_1\right) + \frac{c^3}{2N_2}\left(1 - \cot\theta_2\frac{N_2}{c}h\right)^2\left(\frac{\theta_2}{\sin^2\theta_2} - \cot\theta_2\right), \tag{3.5}$$

$$\begin{aligned} B(h) = &-\frac{c^2}{2}\left(1 - \cot\theta_1\frac{N_1}{c}h\right)^2\left\{1 + \left(\frac{\theta_1}{\sin^2\theta_1} - \cot\theta_1\right)\right. \\ &\left.\times\left[(2\cot^2\theta_1 + 1)\frac{N_1}{c}h - 2\cot\theta_1\right]\right\} + \frac{c^2}{2}\left(1 - \cot\theta_2\frac{N_2}{c}h\right)^2 \\ &\times\left\{1 + \left(\frac{\theta_2}{\sin^2\theta_2} - \cot\theta_2\right)\left[(2\cot^2\theta_2 + 1)\frac{N_2}{c}h - 2\cot\theta_2\right]\right\}, \end{aligned} \tag{3.6}$$

$$C(h) = -\frac{c^2}{2}\left(1 - \cot\theta_1\frac{N_1}{c}h\right)^2 + \frac{c^2}{2}\left(1 - \cot\theta_2\frac{N_2}{c}h\right)^2 - g\Delta\rho h. \tag{3.7}$$

Here

$$\theta_1 = \frac{N_1}{c}(h - H_1), \quad \theta_2 = \frac{N_2}{c}(h - H_2).$$

Multiplying (3.4) by $R(h)h_x$, where function R:

$$R(h) = \frac{1}{A(h)}\exp\left(2\int_0^h \frac{B(h')}{A(h')}\,\mathrm{d}h'\right), \tag{3.8}$$

satisfies the equation $(RA)' = 2BR$, (3.4) reduces to the equation describing zero-energy particle motion in a potential well:

$$\tfrac{1}{2}h_x^2 + U(h) = 0, \tag{3.9}$$

where

$$U(h) = \frac{1}{R(h)A(h)}\int_0^h R(h')C(h')\,\mathrm{d}h'. \tag{3.10}$$

Note that at small h, $C(h) \sim h$ and $U \sim h^2$. To obtain solutions corresponding to solitary waves, it is clearly required that

$$U''(h)|_{h=0} = \left.\frac{C'(h)}{A(h)}\right|_{h=0} < 0. \tag{3.11}$$

An example of the dependence of $U''(0)$ on c is shown on figure 1(a). Also, potential $U(h)$ must have a simple root at some point $h = h_0$, where h_0 is a soliton amplitude

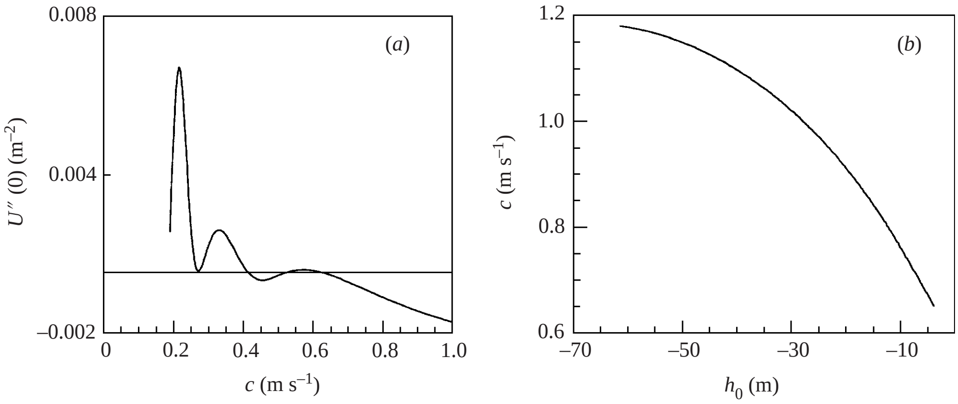

FIGURE 1. (a) The dependence of $U''(0)$ on propagagtion speed c. (b) The dependence of soliton speed on its amplitude.

which generally depends on its velocity. Let us specify the spatial width of the soliton w_x at the level $h = h_0/2$:

$$w_x = 2\int_{h_0}^{h_0/2} \frac{\mathrm{d}h'}{\sqrt{-2U(h')}}. \tag{3.12}$$

The temporal width of the soliton which can be directly compared against point measurements is $w_t = w_x/c$.

4. The case of weakly inhomogeneous layers

Let us consider equation (3.9) for weakly inhomogeneous layers, when

$$N_1|H_1|/c << 1, \quad N_2H_2/c << 1. \tag{4.1}$$

It is easy to see that in the limit $N_1 \to 0$, $N_2 \to 0$ the expressions for coefficients A, B, C, (3.5)–(3.7), reduce to

$$A(h) = \frac{1}{3}\left(-\frac{c^2H_1^2}{h-H_1} + \frac{c^2H_2^2}{h-H_2}\right), \tag{4.2}$$

$$B(h) = \frac{1}{6}\frac{c^2H_1^2}{(h-H_1)^2} - \frac{1}{6}\frac{c^2H_2^2}{(h-H_2)^2}, \tag{4.3}$$

$$C(h) = -\frac{1}{2}\frac{c^2H_1^2}{(h-H_1)^2} + \frac{1}{2}\frac{c^2H_2^2}{(h-H_2)^2} - g\Delta\rho h. \tag{4.4}$$

Hence, in this case $B = A'/2$, $R = 1$, and after simple calculations it follows that

$$U(h) = -\frac{3}{2}\frac{g\Delta\rho(h-H_2)(h-H_1)/c^2 + H_2 - H_1}{H_2^2(h-H_1) - H_1^2(h-H_2)}h^2. \tag{4.5}$$

Equation (3.9) with potential $U(h)$ given by (4.5) coincides with the equation for solitary waves obtained by Choi & Camassa if the Boussinesq approximation is applied to their more general result (see (3.50) in Choi & Camassa 1999) and, as they pointed out, also by Miyata (1985). Properties of corresponding solitons are investigated in detail in Choi & Camassa (1999). Note that the case $H_1 = H_2$, which is degenerate for weakly stratified fluid $A = B = C = 0$, is a regular case in a generic situation.

The case of a homogeneous layer $N = 0$ with a current with a constant shear can be treated similarly to the above. It also leads to an equation like (3.9) with the expression for the potential U being ratio of the third- and the fifth-order polynomials with respect to h (cf. (4.5)).

5. Weakly nonlinear case

Let us assume now that the soliton amplitude is small: $h_0 \to 0$. In this case the width of the soliton increases: $w_x \to \infty$ and its propagation speed tends to the phase speed c_L corresponding to long, linear waves: $c \to c_L$. This speed is determined by solution of the following linear boundary problem:

$$\varphi_{zz} + \frac{N_1^2}{c_L^2}\varphi = 0,\ z < 0; \quad \varphi_{zz} + \frac{N_2^2}{c_L^2}\varphi = 0,\ z > 0; \tag{5.1}$$

$$\varphi_z|_{z=-0}^{z=+0} + \frac{g\Delta\rho}{c_L^2}\varphi(0) = 0; \quad \varphi(H_1) = \varphi(H_2) = 0. \tag{5.2}$$

Eigenfunctions φ are assumed to be normalized according to the condition $\max\varphi = 1$. In the limit $h \to 0$ potential $U(h)$ from (3.11) becomes

$$U(h) \approx \frac{U''(0)h^2}{2} = \frac{C_0'}{2A_0}h^2, \tag{5.3}$$

where lower index 0 indicates the limit $h = 0$. Thus, to obtain $w_x \to \infty$ requires

$$C_0'(c_L) = 0. \tag{5.4}$$

Substituting (3.7) into (5.4) one obtains

$$-c_L N_1 \cot\left(\frac{N_1}{c_L}H_1\right) + c_L N_2 \cot\left(\frac{N_2}{c_L}H_2\right) - g\Delta\rho = 0. \tag{5.5}$$

It is easy to check that the equation for c_L following from the boundary problem (5.1) and (5.2) does coincide with (5.5).

Soliton speed c in the weakly nonlinear case is close to c_L to the accuracy of $O(h_0)$. Taking this into account, we can approximate potential U given by (3.10) and (3.8) to the accuracy of terms of $O(h_0^3)$ as follows:

$$U(h) = \frac{1}{2A_0}\left(\frac{\partial C_0'}{\partial c}\right)_{c=c_L}(c - c_L)h^2 + \frac{C_0''}{6A_0}h^3 + o(h_0^3). \tag{5.6}$$

Let us compare this limit with the result following from the KdV equation. In the latter case

$$h_t + c_L h_x + \alpha h h_x + \beta h_{xxx} = 0, \tag{5.7}$$

where

$$\alpha = \frac{3c_L}{2}\int_{H_1}^{H_2}\varphi_z^3\,\mathrm{d}z\left(\int_{H_1}^{H_2}\varphi_z^2\,\mathrm{d}z\right)^{-1}, \quad \beta = \frac{c_L}{2}\int_{H_1}^{H_2}\varphi^2\,\mathrm{d}z\left(\int_{H_1}^{H_2}\varphi_z^2\,\mathrm{d}z\right)^{-1} \tag{5.8}$$

(see e.g. Ostrovsky & Stepanyants 1989). Considering the stationary solution of (5.7) we find

$$\frac{h_x^2}{2} - \frac{c - c_L}{2\beta}h^2 + \frac{\alpha}{6\beta}h^3 = 0. \tag{5.9}$$

One can check by a direct calculation that

$$-\frac{1}{\beta} = \frac{1}{A_0}\left(\frac{\partial C_0'}{\partial c}\right)_{c=c_L}, \quad \frac{\alpha}{\beta} = \frac{C_0''}{A_0}. \tag{5.10}$$

Comparing (5.9) with (5.6) and taking into account (5.10) demonstrates that the weakly nonlinear solitons defined by (3.4)–(3.7) coincide with KdV solitons for the stratification (1.1).

6. Comparison with the experiment

Large-amplitude internal solitary waves were measured in the COPE experiment (Coastal Ocean Probing Experiment), off the Oregon coast in 1995 (Stanton & Ostrovsky 1998). The stratification at one of the measurement sites can be approximated by a homogeneous sub-surface layer about $H_2 \approx 6$ m thick, a sharp thermocline with overall density jump throughout it of $\Delta\rho \approx 0.003$ and a layer $-H_1 = 144$ m thick with an approximately constant Brunt–Väisälä frequency of $N_1 = 0.012\,\mathrm{s}^{-1}$. A theoretical description of the observed solitons suggested in Stanton & Ostrovsky (1998) was based on a modified KdV equation ('CombKdV'). The solitons calculated according to this model had limiting amplitude $-h_0 \approx 21$ m (since soliton profiles represent a depression their amplitude h_0 is negative) and propagation velocity $c = 0.72\,\mathrm{m\,s}^{-1}$ (a long linear wave has $c = 0.60\,\mathrm{m\,s}^{-1}$). Observed solitons typically had amplitudes in the range $-h_0 \sim 5\,\mathrm{m} - 20\,\mathrm{m}$ and occasionally reached amplitudes of $-h_0 \approx 30$ m. According to radar measurements, typical velocities of solitons were $c = 0.85\,\mathrm{m\,s}^{-1}$. Those values are in a fair agreement with theoretical predictions, especially if one takes into account that the CombKdV model still requires small nonlinearity (i.e. soliton amplitude should be small compared to the vertical scale of stratification, which is $H_2 \approx 6$ m in our case).

In this section, we will apply the model developed in the previous section to solitons observed in COPE. Calculations according to (3.10) and (3.12) with the stratification parameters mentioned above were performed numerically. The dependence of soliton speed on its amplitude is shown on figure 1(b).

The previously mentioned typical propagation speed of $c = 0.85\,\mathrm{m\,s}^{-1}$ corresponds to the amplitude $-h_0 \approx 16$ m, which is in the middle of the range of observed amplitudes. Figure 2(a) shows the dependence of the temporal soliton width w_t on its amplitude. Experimental data from COPE are also shown on the plot by symbols (taken from figure 4(a) of Stanton & Ostrovsky 1998).

One can see that the experimental data on soliton width exhibit some scattering. This could be due to the fact that not all large-amplitude internal waves observed corresponded to pure solitary waves and were, rather, in the evolution stage. Also, small-amplitude solitons are difficult to separate from the background 'noise'. The theoretical curve corresponds to the upper limit of the temporal widths observed and generally is in reasonable agreement with the data for stronger solitons. Figure 2(b) shows a profile corresponding to a soliton with $-h_0 = 26$ m ($c = 0.98\,\mathrm{m\,s}^{-1}$). Experimental points (taken from figure 5 of Stanton & Ostrovsky 1998) are in good agreement with theoretical curve.

Note that our calculations predict a soliton of maximal amplitude $-h_0 \approx 70$ m (close to the limiting amplitude of a soliton due to the Choi & Camassa theory) propagating with a speed of $c = 1.185\,\mathrm{m\,s}^{-1}$.

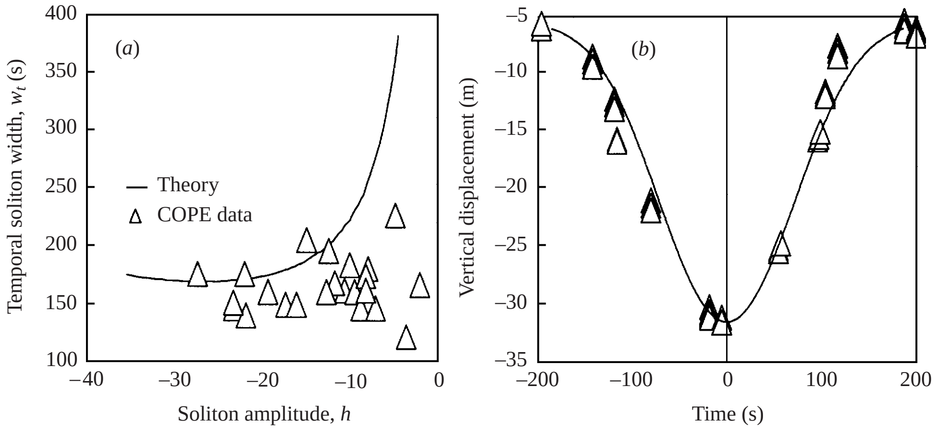

FIGURE 2. (*a*) The dependence of soliton temporal width w_t on its amplitude. (*b*) Soliton profile. Symbols correspond to COPE data, which are taken from figures 4(*a*) and 5 of Stanton & Ostrovsky 1998, respectively.

7. Higher-mode solitons and recirculating core

The solution for a strong stratification case described in the previous section could in principle correspond to higher-mode internal waves as well. It is interesting to investigate the possibility of higher-mode solitons for the case of the COPE stratification. As mentioned, a necessary condition for the existence of a solitary wave is given by (3.11). The dependence of $U''(0)$ on propagation speed c is plotted on figure 1(*a*). Solitons calculated in the previous section corresponded to the region $c > 0.6\,\mathrm{m\,s^{-1}}$. One can see that solitary waves could also exist for velocities between approximately $0.4\,\mathrm{m\,s^{-1}}$ and $0.55\,\mathrm{m\,s^{-1}}$, where $U''(0)$ becomes negative again. This is the case, and numerical calculations show that a soliton with $c = 0.45\,\mathrm{m\,s^{-1}}$ does exist. This soliton can be considered as a second-mode solitary wave, since the overall phase increase throughout the lower layer depth reaches, in this case, $N_1H_1/c \sim 3.84 > \pi$. However, its amplitude appears to be very small and the width rather large: $-h_0 \approx 2.8\,\mathrm{m}$, $w_x \approx 275\,\mathrm{m}$, $w_t \approx 612\,\mathrm{s}$. Such a soliton is difficult to observe in the presence of background internal waves.

A more pronounced higher-mode soliton appears if one decreases the density jump at the boundary between layers to $\Delta\rho = 0.001$, keeping the rest of the stratification parameters the same. Then a soliton with propagation speed $c = 0.39\,\mathrm{m\,s^{-1}}$ has an amplitude of $-h_0 \approx 15.3\,\mathrm{m}$. The overall variation of phase in the lower layer reaches in this case $N_1H_1/c \sim 3.84$. This second-mode soliton also has limiting amplitude which, according to our numerical simulations, is about $-h_0 \approx 17.7\,\mathrm{m}$.

From the general expression for stream function (3.1) with the higher-order terms neglected one can easily see that for the denominator to be non-zero at all x, a soliton profile must be 'squeezed' between appropriate zeros of a function $\sin(N/c)(z - H))$. For weakly nonlinear solitons, when c in this expression can be replaced by c_L, one finds that the soliton profile should be located between zeros of a long-wave linear mode. This can significantly limit soliton amplitude, especially for higher modes. Generally, the following restriction on soliton amplitude holds:

$$N|h_0|/c < \pi. \tag{7.1}$$

Another interesting question is the existence of a soliton recirculating core, i.e. closed flow lines (Tung, Chan & Kubota 1982; Derzho & Grimshaw 1997; Brown &

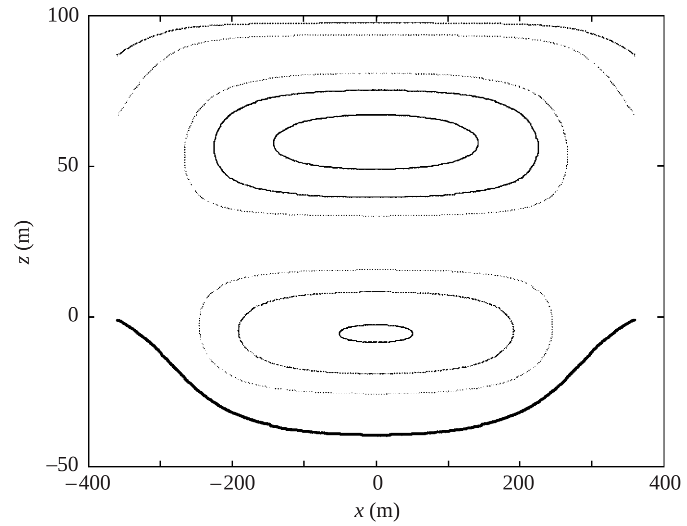

FIGURE 3. Isolines of the stream function.

Christie 1998). The fluid entrained within those lines will propagate with the soliton. An example of such a soliton is shown in figure 3. This soliton is calculated for a hypothetical stratification with the following parameters: $N_1 = 0$, $N_2 = 0.01\,\mathrm{s}^{-1}$, $H_1 = -50\,\mathrm{m}, H_2 = 100\,\mathrm{m}, \Delta\rho = 0$. Soliton speed is $c = 0.235\,\mathrm{m\,s}^{-1}$ and a soliton amplitude is $h_0 = -39.2\,\mathrm{m}$. Analytical investigation of the conditions under which the core appears is difficult. It is required that the equation $\Psi_z + c = 0$ has a root at $x = 0$ (centre of the soliton). Substituting into this equation the representation for Ψ, (3.1), with the higher-order term neglected gives

$$-Nh_0 \cos\left(\frac{N}{c}(z - H)\right) \Big/ \sin\left(\frac{N}{c}(h_0 - H)\right) + c = 0. \qquad (7.2)$$

The necessary condition for this equation to have a root at some $z \in (h_0, H)$ is

$$\left|\frac{Nh_0}{c} \Big/ \sin\left(\frac{N}{c}(h_0 - H)\right)\right| > 1. \qquad (7.3)$$

Thus, the soliton must have sufficiently large amplitude.

However, physical realizability of these solutions is unclear. On one hand, solitons with a recirculating core are not defined uniquely, since an arbitrary function can be substituted into the right-hand side of (2.2) for the stream lines inside the core. Selection of a unique solution should be based on taking account of small viscosity or initial conditions in this case. On the other hand, solutions of (2.2) with a recirculating core corresponding to zero right-hand side calculated numerically by Tung *et al.* (1982) appeared to correspond well to the experimental results by Davis & Acrivos (1967).

8. Conclusions

A theory is presented for the description of the internal solitary waves for a stratification consisting of two layers with a constant Brunt–Väisälä frequency within layers and a density jump at the boundary between them (a ‘2.5-layer model’). The theory makes use of the Boussinesq approximation. In the weakly nonlinear case the solitons obtained reduce to the KdV solitons. For nearly homogeneous layers they coincide with the solitons calculated by Choi & Camassa (within the Boussinesq approximation).

In the general case the solitons calculated here could correspond to higher modes. If the soliton amplitude is large enough, they can also possess a recirculating core, which consists of entrained portions of fluid propagating with the soliton. The physical significance of those solutions is, however, unclear at this point.

The theory is applied to the case of internal wave solitons measured in the COPE experiment, and the results of the theoretical calculations are in reasonable agreement with the measurements.

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